Optimal State-Free, Size-aware Dispatching for Heterogeneous $M/G/$-type systems

Hanhua Feng[1], Vishal Misra[1,2] and Dan Rubenstein[2,1]
Columbia University

[1]Department of Computer Science

[2]Department of Electrical Engineering
Columbia University in the City of New York

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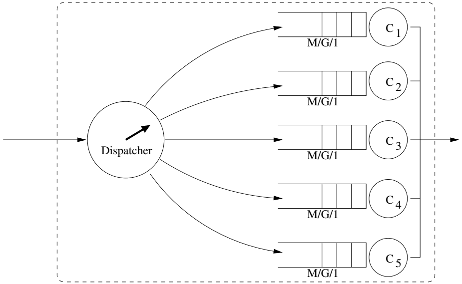
Outline

Dispatching model

- One dispatcher, followed by multiple queues
- Heterogeneous: different server speeds
- FCFS policy for each queue
- Static dispatching strategy



Static strategies

- (Scheduling) Policies: algorithms used by each queue
- (Dispatching) Strategies: algorithms used by dispatcher

A class of static dispatching strategies

- Static: dispatcher is state-free
- Size-aware: dispatcher knows the job size on arrival
- Stochastic: dispatcher may randomly assign jobs

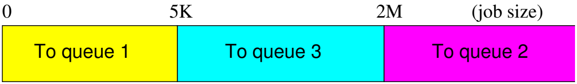
Why static?

- Easy to be implemented
- When collecting dynamic data is hard
- For the baseline of dynamic strategies

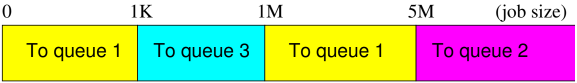
Static strategies

Examples of static strategies

- Random
- Size Interval (SI) [Harchol-Balter 1999]



- Nested Size Interval (NSI) (more generalized than SI)



(Queue 3 is nested in Queue 1)

Analysis with $M/G/-$ model

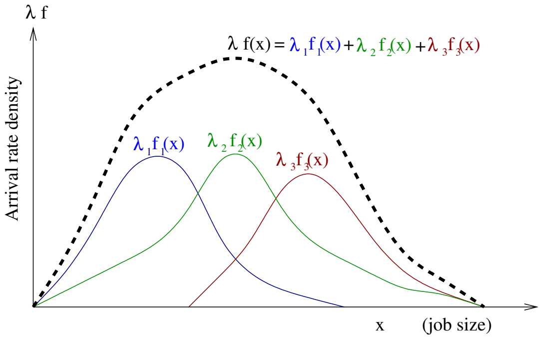
Analysis Assumptions

- Poisson arrival with rate λ
- General known job-size distribution with PDF $f(x)$

Implications

- $\lambda f(x)$ is the arrival rate density function (ARDF)
- Each queue is an $M/G/1$ whose ARDF is $\lambda_i f_i(x)$
- We have $\lambda f(x) = \sum_{i=1}^{n} \lambda_i f_i(x)$.
- The optimal static strategy means the optimal partitioning of function $\lambda f(x)$ to $\lambda_i f_i(x)$ such that the overall mean waiting time is minimized.

Analysis with $M/G/-$ model: example of three queues



- To find $\lambda_1 f_1(x)$, $\lambda_2 f_2(x)$, and $\lambda_3 f_3(x)$ such that $\lambda f(x) = \sum_{i=1}^{3} \lambda_i f_i(x)$ and the mean waiting time is minimized.

Optimal static strategy in homogeneous case

Assume:

- Static strategy: state free, size-aware
- Homogeneous: all servers have the same speed
- $M/G/1$-FCFS queues
- Objective measure: overall per-job mean waiting/response time

Result:

The optimal static strategy is a Size-Interval (SI) strategy.

Optimal static strategy in heterogeneous case

Assume:

- Static strategy: state free, size-aware
- Heterogeneous: servers have different speeds
- $M/G/1$-FCFS queues
- Objective measure: overall per-job mean waiting time

Results:

- The optimal strategy may be a non-SI strategy. (counter-example)
- It is a Nested Size-Interval (NSI) strategy.
- Slower queue can be nested in a faster queue.

Mean waiting time under static strategies

The per-job mean waiting time:

$$E[W] = \frac{1}{2\lambda} \sum_{i=1}^{n} \left[\frac{\lambda_i \omega_i}{c_i(c_i - \rho_i)} \right],$$

where

- λ_i is arrival rate of queue i,
- ρ_i is the (first-order) load of queue i,
- $\omega_i = \lambda \int_0^\infty t^2 f_i(t) dt$ is called "second-order load" of queue i,
- c_i is the capacity (processing speed) of queue i.

Mean waiting time under static strategies (cont'd)

Mean waiting time

$$E[W] = \frac{1}{2\lambda} \sum_{i=1}^{n} \left[\frac{\lambda_i \omega_i}{c_i(c_i - \rho_i)} \right]$$

Objective: minimizing $E[W]$

- Recall that: partitioning $\lambda f(\cdot)$ to a sum of $\lambda_i f_i(\cdot)$'s
- Objective $E[W]$ depends on λ_i, ρ_i and ω_i.
- λ_i, ρ_i and ω_i are respectively zeroth-, first-, and second-order moments of $\lambda_i f_i(\cdot)$.

Proof for homogeneous case ($c_1 = c_2 = \cdots$)

First prove for two queues, then extend to multiple queues.

For two queues: two cases

- Case 1: their loads are severely unbalanced: either of
  - $\frac{\lambda_1}{c_1(c_1-\rho_1)} > \frac{\lambda_2}{c_2(c_2-\rho_2)}$ and $\frac{\omega_1}{c_1(c_1-\rho_1)} > \frac{\omega_2}{c_2(c_2-\rho_2)}$
  - $\frac{\lambda_1}{c_1(c_1-\rho_1)} < \frac{\lambda_2}{c_2(c_2-\rho_2)}$ and $\frac{\omega_1}{c_1(c_1-\rho_1)} < \frac{\omega_2}{c_2(c_2-\rho_2)}$
- Case 2: otherwise.

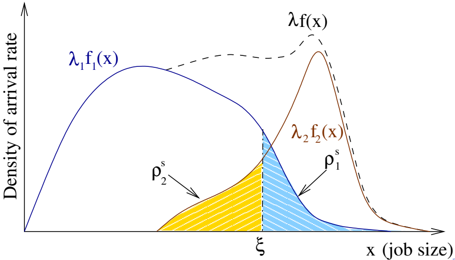
Case 1: Severely unbalanced

- Transfer some jobs (of any size) from the high-loaded queue to the other.
- $E[W]$ is lower.
- Repeat doing so until Case 2.

Proof for homogeneous case (cont'd)

Case 2: not severely unbalanced

- Find a threshold such that shaded areas have same ρ.
- Swapping shaded areas yields an SI strategy.
- We show that $E[W]$ is lower.



Proof for heterogeneous case ($c_1 < c_2 < \cdots$)

For two queues: $c_1 < c_2$

- Case 1: their loads are severely unbalanced: either of
  - Case 1a: $\frac{\lambda_1}{c_1(c_1-\rho_1)} > \frac{\lambda_2}{c_2(c_2-\rho_2)}$ and $\frac{\omega_1}{c_1(c_1-\rho_1)} > \frac{\omega_2}{c_2(c_2-\rho_2)}$
  - Case 1b: $\frac{\lambda_1}{c_1(c_1-\rho_1)} < \frac{\lambda_2}{c_2(c_2-\rho_2)}$ and $\frac{\omega_1}{c_1(c_1-\rho_1)} < \frac{\omega_2}{c_2(c_2-\rho_2)}$
- Case 2: otherwise.

Cases 1b and Case 2

- Use the same arguments as in a homogeneous system.

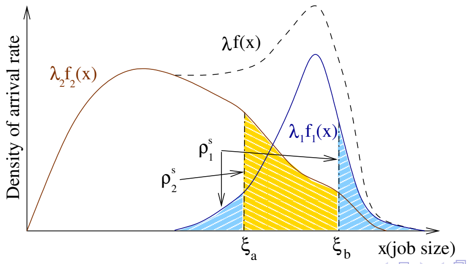
Case 1a

- Previous argument fails.

Proof for heterogeneous case (cont'd)

Case 1a (Slower queue has severely higher load)

- find two thresholds: shaded areas have same λ's and ρ's.
- Swapping shaded areas yields an NSI strategy
- We show that $E[W]$ gets lower.



Remaining issues for finding an optimal static strategy

Homogeneous case

- To find the optimal thresholds of size intervals.
- Mapping (which queue gets which interval) is irrelevant.

Heterogeneous case

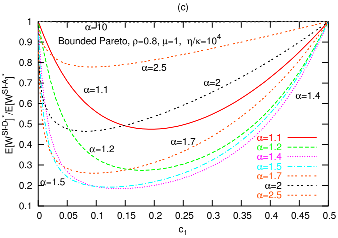
- Nested size intervals are more complicated.
- Mappings between queues and intervals matter.

What is the best mapping in heterogeneous case?

- No fixed rules(e.g. slower queue gets the interval of shorter jobs)
- Depending on job size distributions

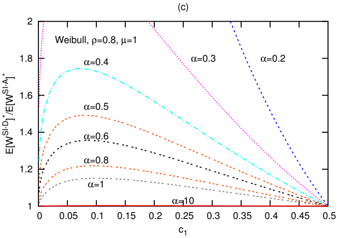
Mapping of intervals to queues: examples

- Only SI strategy (more simpler than NSI) and two queues
- Compare two mappings (ascending and descending) with corresponding optimal thresholds



Bounded Pareto

Best: faster server gets short jobs



Weibull

Best: faster server gets long jobs

(Y-axis: ratio between optimal mean response times of ascending and descending mapping. X-axis: speed fraction of the slower queue)

Mapping of intervals to queues: examples (cont'd)

- Two queues, SI strategy
- Only two mappings
  - Ascending mapping: faster server gets interval of long jobs
  - Descending mapping: faster server gets interval of short jobs
- For each mapping, an optimal threshold (or load partitioning) can be found.
- Log-normal distribution is mapping-invariant here (two mappings obtains the same optimal) response time.
- All distributions such that

$$m(x) = E[X] - m\left(\frac{\psi}{x}\right), \text{ where } m(x) = \int_0^x t dF(t), \psi\text{:constant}$$

is mapping-invariant (for two queues and SI strategy).

Conclusion

- For FCFS, optimal static strategy is an NSI.
- For FCFS homogeneous queues, optimal static strategy is an SI.
- Other scheduling policies can be used on each queue.
- For PS, size information is useless (w.r.t. mean response time).
- Dynamic case is more complicated. [Whitt 1984]
- It is difficult to find the best mapping.