Linear Approximation of Optimal Attempt Rate in Random Access Networks

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***Abstract*— While packet capture has been observed in real implementations of wireless devices randomly accessing shared channels, fair rate control algorithms based on accurate channel models that describe the phenomenon have not been developed. In this paper, using a general physical channel model, we develop the equation for the optimal attemp rate to maximize the aggregate log utility. We use the least squares method to approximate the equation to a linear function of the attempt rate. Our analysis on the approximation error shows that the linear function obtained is close enough to the original with the square of the residuals more than 0.9.**

I. Obtaining Critical Point to Maximize Log Utilities in Random Access Networks

We develop an algorithm to obtain optimal rates for log utility fairness [1] using the fixed-point iteration. Let S be the aggregate log utility. The aggregate utility S in slotted-Aloha systems is $\sum_i \log(f_i q_i)$, where f_i is the attempt rate of node i, q_i is the success probability of transmissions from node i. For CSMA/CA systems with a single carrier-sensing ranges, S is given by $\sum_i \log(\frac{f_i q_i p_{TX}}{(1-p_{TX})T_{SL}+p_{TX}T_{TX}})$, where p_{TX} is the probability where any nodes in the network transmit at a time slot, which is simply given by $p_{TX} = 1-\prod_{j\in N}(1-f_j)$,$T_{SL}$ and T_{TX} are the length of the time slot and transmission time.

The optimal attempt rate allocation makes the differentiation of the aggregate utility equal to zero for all i. Thus,

$$\frac{\partial S}{\partial f_i} = \frac{1}{f_i} - \frac{1}{1-f_i}\sum_{j\in N}(\frac{T_{TX}}{(1-p_{TX})T_{SL}+p_{TX}T_{TX}} - \frac{q_{j|i}}{q_j})$$
$$= \frac{1}{f_i} - \underbrace{\frac{|N|}{G-(1-f_i)}}_{A} - \underbrace{\frac{|N| - \sum_{j\in N}\frac{q_{j|i}}{q_j}}{1-f_i}}_{B} = 0, \quad (1)$$

where G is $\frac{T_{TX}}{(T_{TX}-T_{SL})\prod_{j\neq i}(1-f_j)}$. Note that Part A in Equation 1 is omitted if any attempt rates of the nodes equals to 1 or $T_{TX} = T_{SL}$.

Especially, for node i, the optimal attempt rate f_i satisfies the following:

$$\frac{1-f_i}{f_i} = \sum_j(\frac{T_{TX}}{(1-p_{TX})T_{SL}+p_{TX}T_{TX}} - \frac{q_{j|i}}{q_j}). \quad (2)$$

Let F^* be the optimal attempt rate vector satisfying Equation 1 and function $g(F^*) = F^*$. Since function g is continuous and maps a rate vector to another rate vector, g has a fixed point (Brouwer's fixed point theorem [2]). We can further show that g converges to the fixed point. Thus, if we know g, F^* is obtained by continuously applying g.

Now, we formulate a function that returns f_i satisfying Equation 1, given f_j for $j \neq i$. Since p_{TX} and q_j need f_i to compute, Equation 1 is not easy to solve. To get the function to compute f_i, we first show that Part A in Equation 1 is approximated to a linear function of f_i as follows:

$$\text{Part A} \approx |N|(f_i\mu(\prod_{j\neq i}(1-f_j)) + \nu(\prod_{j\neq i}(1-f_j))). \quad (3)$$

We obtain μ and ν by applying the least squares method. Given f_j ($j \neq i$), we uniformly sample K points from the curve $1/(G-(1-f_i))$, which is a function of f_i, and find a linear function that closely approximates the sampled data to minimize the sum of the squares of the residuals between points generated by the function and corresponding sampled points. The computation time of this approximation is $O(K)$.

Part B in Equation 1 is approximated by:

$$\text{Part B} = \sum_j \frac{q_{j|\bar{i}} - q_{j|i}}{f_i q_{j|i} + (1-f_i)q_{j|\bar{i}}}$$
$$\approx \sum_j(\theta(q_{j|i}, q_{j|\bar{i}})f_i + \phi(q_{j|i}, q_{j|\bar{i}})). \quad (4)$$

After linear approximation, we have a quadratic formula for f_i from Equation 1 as follows:

$$\frac{1}{f_i} = af_i + b, \quad (5)$$

where $a = |N|\mu(\prod_{j\neq i}(1-f_j)) + \sum_j \theta(q_{j|i}, q_{j|\bar{i}})$, $b = |N|\nu(\prod_{j\neq i}(1-f_j)) + \sum_j \phi(q_{j|i}, q_{j|\bar{i}})$.

From the quadratic formula, f_i is finally given by:

$$f_i = \begin{cases} \frac{-b+\sqrt{b^2+4a}}{2a} & \text{(if } a > 0\text{) and} \\ \min(\frac{1}{b}, 1) & \text{(if } a = 0\text{),} \end{cases}$$

where $\min(\infty, 1) = 1$. Assuming q_j and $q_{j|i}$ are known, the total computation time of f_i is $O(K)$.

It is easy to see that G in Part A of Equation 1 is greater than 1. A linear approximation to Part A with f_i is prone to larger error as G gets closer to 1. We, however, show the approximation error is small enough. The value of

$\prod_{j \neq i}(1-f_j)$ is maximized when all f_j is the minimum. From Equation 2, the possible minimum of f_i is obtained when $q^*_{j|i} = 0$. Using 802.11 operation parameters to compute T_{TX} and T_{SL}, G is at least 1.18 for all $|N| \geq 2$ sending 512-byte packets at 54 Mbps. For all $|N| \geq 5$, G is more than 1.30. The larger $|N|$ is, the bigger G we have. With $|N| = 2$, the square of the residuals is around 0.875008831 for f_i in the range of 0 to 0.5. When $|N| = 5$, the square value is beyond 0.927489457. For $f_i > 0.5$ and $|N| \geq 2$, the square value is more than 0.976722922. Note that the square of the residuals is an indicator of how well the linear equation fits. It ranges in value from 0 to 1 and the value 0.927489457 indicates that there is an close correlation in the estimated and actual value of Part A.

To improve the accurate further, we repeat the approximation with different intervals and obtain f_i. That is, after f_i is obtained, we find another best-fitting line for a segment of the curve in an interval around the value of f_i. Then, we again compute a new value of f_i satisfying Equation 1. Repeating this process several times, we attain the accurate value of f_i.

In approximation for Part B, the square of the residuals is 0.968194017 for f_i in the range of 0 to 0.5. Since $q_{j|\bar{i}}$ and $q_{j|i}$ are not a function of f_i, Part B is continuous between $1/q_{j|i}$ and $1/q_{j|\bar{i}}$. It is trivial that the shorter distance between the two end points, the more like a line the graph looks. We can claim that the square of the residuals is maximized when $q_{j|\bar{i}}$ and $q_{j|i}$ are farthest away from each other, where $q_{j|\bar{i}} = 1$ and $q_{j|i} = 0$ (because $q_{j|\bar{i}} \geq q_{j|i}$). Note that the optimal value of f_i is typically less than 0.5 in CSMA/CA and slotted-Aloha systems. However, for $f_i > 0.5$, we can compute accutate μ, ν, θ and ϕ by repeating the approximation with proper intervals.

References

[1] F. Kelly, A. Maulloo, and D. Tan, “Rate control in communication networks: shadow prices, proportional fairness and stability,” in *Journal of the Operational Research Society*, vol. 49, 1998.

[2] W. S. Massey, *Algebraic Topology: An Introduction*. Springer-Verlag, 1977, vol. 56.